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Random Sturm–Liouville operators with point interactions

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Abstract

We study invariance for eigenvalues of selfadjoint Sturm–Liouville operators with local point interactions. Such linear transformations are formally defined by

$$H_\omega := -\frac{d^2}{dx^2} + V(x) + \sum_{n\in I} \omega(n)\delta\left(x - x_n\right)$$

or similar expressions with δ' instead of δ. In a probabilistic setting, we show that a point is either an eigenvalue for all ω or only for a set of ω's of measure zero. Using classical oscillation theory it is possible to decide whether the second situation happens. The operators do not need to be measurable or ergodic. This generalizes the well known fact that for ergodic operators a point is eigenvalue with probability zero.

KEYWORDS
eigenvalue, point interactions, random operator, random Sturm–Liouville operator, singular perturbation

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1 | INTRODUCTION

This work is about point spectra of selfadjoint Sturm–Liouville operators with δ, δ'-interactions. These are defined by expressions of the form

$$H_\omega := -\frac{d^2}{dx^2} + V(x) + \sum_{n\in I} \omega(n)\delta\left(x - x_n\right)$$

or with δ' instead of δ. There are several ways to introduce this objects. They can be constructed by using form methods, see [16] or by adding boundary conditions as in [11], for example. Here we shall use the approach developed in [5] which generalizes Sturm–Liouville classical theory to include local point interactions. This has the advantage that selfadjointness, the Weyl alternative and related results can be established along the lines of a well known theory. For a detailed study of this field, including many solvable models in quantum mechanics as well as an extensive list of references see the monograph [4].

The relations between the operators and their spectra, have deep consequences in several areas of functional analysis, scattering theory, localization problems, dynamic behavior of quantum systems, differential and integral equations, matrix theory and so on. We shall focus on the point spectrum and consider operators generated by δ or δ' interactions with one common eigenvalue. This can be regarded as an inverse spectral problem, where given a point $E \in \mathbb{R}$ one tries to characterize the sequences ω for which E belongs to the point spectra of the operators H_ω. Placing our operators in a random environment, we are able to give the characterization of operators sharing the same eigenvalue in a probabilistic setting.

In the random situation we consider here, the ω associated to H_ω is a stochastic process and each $\omega(n)$ a random variable with continuous (maybe singular) probability distribution. Our operators H_ω do not have to be measurable (see Definition 4.13) and ω does not have to be a stationary metrically transitive random field or ergodic, see Section 9.1 [7]. For metrically transitive random operators it is well known that the probability for a given $E \in \mathbb{R}$ to be an eigenvalue is zero (see [15, Section 4.3, Corollary 1], [20, Theorem 2.12]). If we do not have this condition, in principle any situation could be possible. We show that for random operators with point interactions, the following alternative holds: a point is either an eigenvalue for all ω or only for a set of ω's of measure zero. This will be a consequence of the fact that a similar behavior for seladjoint extensions of symmetric operators with defect indices (1,1) takes place. To decide which of these situations happens we were able to use classical oscillation theory, exploiting the relation between the zeros of eigenfunctions and the placement of the points interactions.

This work is organized as follows. In Section 2 we consider rank one singular perturbations defined as selfadjoint extensions of symmetric operators with deficiency indices (1,1). It is shown that eigenvalues of these selfadjoint extensions either disappear or remain fixed for every extension. Section 3 deals with singular rank one perturbations generated by delta functionals, we introduce the basic facts that will be used in the next section. In Section 4 we study the random case. Here one main result is Theorem 4.4 which gives a characterization of the ω's such that H_ω share an eigenvalue. Subsection 4.1 considers zeros of eigenfunctions belonging to the operator without point interactions. It is proven in particular, that nonoscillatory behavior implies the family H_ω's has a common eigenvalue for a set of ω's of measure zero. Analogous results hold if the interactions are placed close enough. In subsection 4.2 measurable operators are introduced. Finally, in Section 5 we study operators with δ'-interactions and show that similar results to the ones for δ hold. Another way to obtain the same results can be found in a previous version of this work [8].

We denote as usual by $\mathbb{R}, \mathbb{C}$ the real and complex numbers, by $L^1(J) := \{f : J \to \mathbb{C} : \int_J |f| < \infty\}$ the integrable functions, $L^1_{loc}(J) := \{f \in L^1(\tilde{J}) : \tilde{J} \subseteq J, \tilde{J} \text{ a closed interval}\}$ the local integrable functions, $C_0^\infty(J)$ the infinitely differentiable functions with compact support on J and the eigenvalues of an operator L by $\sigma_p(L)$. Given a transformation $T : V_1 \to V_2$, where V_1 and V_2 are vectorial spaces, we use the notation $\text{Kern}(T) := \{v \in V_1 : T(v) = 0\}$.

2 | STABILITY OF EIGENVALUES OF SINGULAR RANK ONE PERTURBATIONS

Let $\mathcal{H}$ be a Hilbert space, let $A : \mathcal{H} \to \mathcal{H}$ be a selfadjoint operator with dense domain $D(A)$ and let $\varphi : D(A) \to \mathbb{C}$ be a linear functional. Let $\dot{A}$ be the restriction of A to

$$D(\dot{A}) := \text{Kern}(\varphi) = \{\psi \in D(A) : \varphi(\psi) = 0\}. \tag{2.1}$$

Rank one perturbations are formally defined by

$$A_\alpha = A + \alpha\langle\varphi, \cdot\rangle\varphi$$

and they correspond to selfadjoint extensions of the symmetric operator $\dot{A}$. See for example [2–4, 13, 17].

Our first task will be to study the stability of the eigenvalues of the operators T_θ to be introduced bellow. This will be the key to understand the behavior of the eigenvalues of selfadjoint extensions of $\dot{A}$. For the next result compare the brilliant analysis done by Donoghue in [9, Section 2].

Let $H : \mathcal{H} \to \mathcal{H}$ be a symmetric operator with domain $D(H)$ and defect indices (1,1). Let $T_\theta : \mathcal{H} \to \mathcal{H}$ denote the selfadjoint extensions of H with $0 \leq \theta < 2\pi$ which are given by von Neumann's theory. Then we have the following alternative:

Theorem 2.1. *Assume E is an eigenvalue of T_β for some $\beta \in [0, 2\pi)$, then one of the following happens:*

i) E is an eigenvalue of T_θ for all $\theta \in [0, 2\pi)$.
ii) E is not an eigenvalue of T_θ for $\theta \neq \beta$.

Case i) holds if and only if the eigenvector associated to E is on $D(H)$.

Proof. It will be enough to proof that if $i)$ does not hold then $ii)$ must hold.

Assume there exists $\alpha \in [0, 2\pi)$ such that $E \notin \sigma_p(T_\alpha)$ and suppose that E is an eigenvalue of T_γ for some $\gamma \neq \beta$. We shall prove that this is not possible and therefore conclude that $\gamma = \beta$. So let us assume there exist $\varphi \in D(T_\beta)$ and $\psi \in D(T_\gamma)$ such that

$$T_\beta \varphi = E\varphi \qquad \text{and} \qquad T_\gamma \psi = E\psi. \tag{2.2}$$

From (2.2) and since $T_\beta, T_\gamma \subset H^*$ we have that $\varphi, \psi \in \mathrm{Kern}(H^* - E)$. We claim that $\dim(\mathrm{kern}(H^* - E)) = 1$. To see this, let $f \in \mathrm{Kern}(H^* - E)$ and let g be any function in $D(H)$, then

$$\begin{aligned}
\left\langle (T_\alpha - E)(T_\alpha - i)^{-1} f, (H + i)g \right\rangle &= \left\langle f + (i - E)(T_\alpha - i)^{-1} f, (H + i)g \right\rangle \\
&= \langle f, (H + i)g\rangle + \left\langle (i - E)(T_\alpha - i)^{-1} f, (H + i)g \right\rangle \\
&= \langle f, (H - E + E + i)g\rangle + \left\langle (i - E)(H^* - i)(T_\alpha - i)^{-1} f, g \right\rangle \\
&= \langle f, (H - E)g\rangle + \langle f, (E + i)g\rangle + \langle (i - E)f, g\rangle \\
&= \langle f, (H - E)g\rangle \\
&= \langle (H^* - E)f, g\rangle = 0.
\end{aligned}$$

Then $(T_\alpha - E)(T_\alpha - i)^{-1} : \mathrm{Kern}(H^* - E) \to \mathrm{Kern}(H^* - i)$ is a well defined, injective and linear map. Therefore

$$\dim(\mathrm{Kern}(H^* - E)) = \dim(\mathrm{Kern}(H^* - i)) = 1 \tag{2.3}$$

proving the claim.

From the von Neumann's extension theory of symmetric operators (see for example [23, Theorem 13.10] or [25, Theorem 8.12]) we have for φ and ψ the following representations:

$$\varphi = \varphi_0 + c_1 u_i + c_1 e^{i\beta} u_{-i} \in D(T_\beta) \tag{2.4}$$

and

$$\psi = \psi_0 + c_2 u_i + c_2 e^{i\gamma} u_{-i} \in D(T_\gamma) \tag{2.5}$$

with $\varphi_0, \psi_0 \in D(H)$, $c_1, c_2 \in \mathbb{C}$ and $u_{\pm i} \in \mathrm{Kern}(H^* \mp i)$. From (2.2) and (2.3) we can consider $\varphi = \psi$. We have

$$\varphi_0 + c_1 u_i + c_1 e^{i\beta} u_{-i} = \psi_0 + c_2 u_i + c_2 e^{i\gamma} u_{-i}.$$

Since the sums are direct then $\varphi_0 = \psi_0$ and $(c_1 - c_2)u_i + (c_1 e^{i\beta} - c_2 e^{i\gamma})u_{-i} = 0$. Since u_i and u_{-i} are linearly independent $c_1 = c_2$. If $c_1 = c_2 = 0$, then $\varphi, \psi \in D(H)$ and ψ is an eigenvector of T_α with eigenvalue E which contradicts our hypothesis. If $c_1 = c_2 \neq 0$, then $e^{i\beta} = e^{i\gamma}$. Thus $\beta = \gamma$ and *ii*) holds.

If the eigenvector associated to E is on $D(H)$ then E is an eigenvalue of T_θ for all $\theta \in [0, 2\pi)$, since $H \subset T_\theta$. To see the other direction, assume E is an eigenvalue of T_θ for all $\theta \in [0, 2\pi)$ and choose $\theta_0 \neq \beta$. Let φ as in (2.2), if $\varphi \notin D(H)$, from the representation (2.4) we have $c_1 \neq 0$ and then follows as above that $\beta = \theta_0$ which is a contradiction. Therefore if E is an eigenvalue of T_θ for all $\theta \in [0, 2\pi)$ the eigenvector associated to E is on $D(H)$. □

The next two lemmas are essentialy reformulations of [2, Lemma 2.1] or [3, Lemma 1.2.3]. They will allow us to prove that $\dot{A}$ is a densely defined symmetric operator with defect indices (1,1).

Let $\varphi : D \subset \mathcal{H} \to \mathbb{C}$ be a linear functional defined on a dense set D of a Hilbert space $\mathcal{H}$.

Lemma 2.2. *If φ is discontinuous in D, then* Kern(φ) *is dense in $\mathcal{H}$.*

Proof. Since φ is discontinuous there exists a sequence $\{x_n\} \subset D$ such that $\|x_n\| = 1$ and $|\varphi(x_n)| \to \infty$ as $n \to \infty$. Take $y \in D$ and write

$$y = y - \frac{\varphi(y)}{\varphi(x_n)} x_n + \frac{\varphi(y)}{\varphi(x_n)} x_n.$$

Suppose $x \in \mathrm{Kern}(\varphi)^{\perp}$, hence since $y - \frac{\varphi(y)}{\varphi(x_n)} x_n \in \mathrm{Kern}(\varphi)$ we have the inequality

$$|\langle x, y\rangle| = \left|\left\langle x, \frac{\varphi(y)}{\varphi(x_n)} x_n \right\rangle\right| \leq \|x\| \frac{|\varphi(y)|}{|\varphi(x_n)|} \to 0 \qquad \text{as } n \to \infty.$$

Therefore $D \subset \mathrm{Kern}(\varphi)^{\perp\perp} = \overline{\mathrm{Kern}(\varphi)}$. Since D is dense $\mathcal{H} = \overline{D} \subset \overline{\mathrm{Kern}(\varphi)}$ and therefore $\overline{\mathrm{Kern}(\varphi)} = \mathcal{H}$, so $\mathrm{Kern}(\varphi)$ is dense. □

For the next lemma let A, $\dot{A}$ and φ be as in (2.1).

Lemma 2.3. *Assume φ is discontinuous in $D(A)$ with the norm of $\mathcal{H}$. Let $l : \mathcal{H} \to \mathbb{C}$ be the functional defined as $l(\psi) := \varphi\big((A+i)^{-1}\psi\big)$. If l is continuous on $\mathcal{H}$, then $\dot{A}$ is a densely defined symmetric operator with deficiency indices (1,1).*

Proof. From Lemma 2.2 follows that the domain $D(\dot{A}) := \mathrm{Kern}(\varphi)$ is dense. Now for $\gamma \in \mathcal{H}$, we have

$$l(\gamma) = 0 \Leftrightarrow (A+i)^{-1}\gamma \in \mathrm{Kern}(\varphi) = D(\dot{A}) \Leftrightarrow \gamma \in \mathrm{Rang}(\dot{A} + i).$$

Therefore

$$\mathrm{Kern}(l) = \mathrm{Rang}(\dot{A} + i) \tag{2.6}$$

and this set is closed by the continuity of l.

Now $\mathrm{Kern}(l) \neq \mathcal{H}$ because if $l(\psi) = 0$, for all $\psi \in \mathcal{H}$, from Equation (2.6) and the basic criterion for selfadjointness we conclude that $\dot{A}$ is selfadjoint and therefore $\dot{A} = A$. That would mean that $\mathrm{Kern}(\varphi) = D(A)$ and therefore φ continuous in $D(A)$ which is a contradiction to the hypothesis. Since l is continuous and linear, by the Riesz lemma there exists $h \in \mathcal{H}$, $h \neq 0$, such that $\langle h, \cdot\rangle = l(\cdot)$.

Taking into account that $\mathrm{Rang}(\dot{A} + i)$ is closed and therefore equal to $\mathrm{Kern}\Big(\big(\dot{A}\big)^* - i\Big)^{\perp}$, we have

$$\{\gamma : \langle h, \gamma\rangle = 0\} = \mathrm{Kern}(l) = \mathrm{Kern}\Big(\big(\dot{A}\big)^* - i\Big)^{\perp}.$$

It follows that

$$\{ch : c \in \mathbb{C}\} = \{\gamma : \langle h, \gamma\rangle = 0\}^{\perp} = \mathrm{Kern}\Big(\big(\dot{A}\big)^* - i\Big).$$

Therefore, $\dim\mathrm{Kern}\Big(\big(\dot{A}\big)^* - i\Big) = 1$. Since $\dot{A}$ has selfadjoint extensions, the deficiency indices are equal and $\dim\mathrm{Kern}\big(\big(\dot{A}\big)^* + i\big) = 1$. □

As mentioned above, the selfadjoint extensions of $\dot{A}$ give a precise meaning to the singular rank one perturbations formally given by

$$A_\alpha = A + \alpha\langle\varphi, \cdot\rangle\varphi.$$

Then applying Lemmas 2.2 and 2.3 to $\dot{A}$ and Theorem 2.1 to these extensions, we can formulate the following result:

Theorem 2.4. *A point is either eigenvalue for all the rank one singular perturbations of a given selfadjoint operator or for at most one of them.*

Remark 2.5. It can be proven that when $\varphi \in \mathcal{H}$ the same statement is true.

3 | STURM–LIOUVILLE OPERATORS WITH δ-POINT INTERACTIONS

3.1 | Preliminary definitions

Let $-\infty \leq a < b \leq \infty$, let $V \in L^1_{loc}(a,b)$ be a real valued function. Fix a discrete set M of points accumulating at most at a or b, $M := \{x_n\}_{n \in I} \subset (a,b)$ where $I \subseteq \mathbb{Z}$ and let $\{\alpha_n\} \subset \mathbb{R}$. We set $\alpha = \alpha_{n_0}$ and consider the formal differential expressions

$$\tau := -\frac{d^2}{dx^2} + V, \tag{3.1}$$

$$\tau_{\alpha,M} := -\frac{d^2}{dx^2} + V(x) + \sum_{n \in I \setminus \{n_0\}} \alpha_n \delta(x - x_n) + \alpha\delta(x - x_{n_0}).$$

The maximal operator $T_{\alpha,M}$ corresponding to $\tau_{\alpha,M}$ is defined by

$$T_{\alpha,M} f = \tau f,$$

$$D(T_{\alpha,M}) = \{f \in L^2(a,b) : \ f,\ f' \text{ abs. cont. in } (a,b)\setminus M, -f'' + Vf \in L^2(a,b),$$
$$f(x_n+) = f(x_n-),\ f'(x_n+) - f'(x_n-) = \alpha_n f(x_n),\ \forall n \in I\}.$$

We make the next definitions following [5].

Definition 3.1. Given $g \in L^1_{loc}(a,b)$ and $z \in \mathbb{C}$, we call f a *solution* of $(\tau_{\alpha,M} - z)f = g$ if f and f' are absolutely continuous in $(a,b)\setminus M$ with $-f'' + Vf - zf = g$ and $f(x_n+) = f(x_n-)$, $f'(x_n+) - f'(x_n-) = \alpha_n f(x_n)$, for all $n \in I$.

Definition 3.2. We define the *Wronskian of two solutions* u_1 and u_2 of $(\tau_{\alpha,M} - z)f = 0$ as

$$W_x(u_1, u_2) = u_1(x+)u_2'(x+) - u_1'(x+)u_2(x+).$$

Note that the Wronskian is constant in (a,b), see [5, Lemma 4.2].

Definition 3.3. For $f, g \in D(T_{\alpha,M})$ we define the *Lagrange bracket* by

$$[f,g]_x = \overline{f(x+)}g'(x+) - \overline{f'(x+)}g(x+).$$

The limits $[f,g]_a = \lim_{x\to a+}[f,g]_x$ and $[f,g]_b = \lim_{x\to b-}[f,g]_x$ exist. See [5, Theorem 2.2].
A solution of $(\tau_{\alpha,M} - z)f = 0$ is said to *lie right (left) in* $L^2(a,b)$, if f is square integrable in a neighborhood of b (a).

Definition 3.4.

i) $\tau_{\alpha,M}$ is in the *limit circle case* (lcc) at b if for every $z \in \mathbb{C}$ all solutions of $(\tau_{\alpha,M} - z)f = 0$ lie right in $L^2(a,b)$.
ii) $\tau_{\alpha,M}$ is in the *limit point case* (lpc) at b if for every $z \in \mathbb{C}$ there is at least one solution of $(\tau_{\alpha,M} - z)f = 0$ not lying right in $L^2(a,b)$.

The same definition applies to the endpoint a.

According to the Weyl Alternative, see [5, Theorem 4.4], we have always either *i*) or *ii*).

Consider the selfadjoint restriction $H_{\alpha,M}$ of $T_{\alpha,M}$ in $L_2(a,b)$ defined as

$$H_{\alpha,M}f = \tau f \tag{3.2}$$

$$D(H_{\alpha,M}) = \left\{ f \in D(T_{\alpha,M}) : \begin{array}{l} [v,f]_a = 0 \text{ if } \tau_{\alpha,M} \text{ lcc at } a, \\ [w,f]_b = 0 \text{ if } \tau_{\alpha,M} \text{ lcc at } b \end{array} \right\}.$$

Where v and w are non-trivial real solutions of $(\tau_{\alpha,M} - \lambda)v = 0$ near a and near b respectively, $\lambda \in \mathbb{R}$. See [5, Theorem 5.2].

For the next definition we take $\alpha = 0$.

Definition 3.5. Let us define the *Green's function* $G(x,y;z)$ for $H_{0,M}$ as

$$G(x,y;z) := \begin{cases} W(u_b,u_a)^{-1} u_a(x,z)u_b(y,z) & \text{if } x \le y, \\ W(u_b,u_a)^{-1} u_b(x,z)u_a(y,z) & \text{if } x > y, \end{cases} \tag{3.3}$$

where u_a and u_b are solutions of $(\tau_{0,M} - z)u = 0$ (see Definition 3.1) with $[v,u_a]_a = 0$ if $\tau_{0,M}$ lcc at a and $[w,u_b]_b = 0$ if $\tau_{0,M}$ lcc at b.

According to [5, Theorem 5.2 b)] we have

$$\left((H_{0,M} - z)^{-1} g\right)(x) = \int_a^b G(x,y;z)g(y)\,dy. \tag{3.4}$$

Definition 3.6. We say $\tau_{\alpha,M}$ is *regular at a* if a is finite, $V \in L^1_{loc}[a,b)$ and a is not an accumulation point of M. The same definition applies to the endpoint b.

If $\tau_{\alpha,M}$ is regular at a, then $\tau_{\alpha,M}$ is lcc at a and the condition $[v,f]_a = 0$ can be replaced by

$$f(a)\cos\psi + f'(a)\sin\psi = 0$$

for $\psi \in [0,\pi)$. The same holds for b.

3.2 | Behavior of the eigenvalues of operators with δ-point interactions

Let us take from now on $\varphi : D(H_{0,M}) \to \mathbb{C}$ given by $\varphi(f) := \delta_{x_{n_0}}(f) = f(x_{n_0})$.

Lemma 3.7.

i) The functional φ is not continuous in $D(H_{0,M})$ with the norm of $L^2(a,b)$.

ii) The functional $l : L^2(a,b) \to \mathbb{C}$ defined as $l(f) := \varphi\left((H_{0,M} + i)^{-1} f\right)$ is continuous.

Proof.

i) Take $\epsilon > 0$ such that $I_\epsilon \cap M = \{x_{n_0}\}$, where $I_\epsilon = [x_{n_0} - \epsilon, x_{n_0} + \epsilon]$. Let $F \in C_0^\infty(-\epsilon,\epsilon)$ such that $F(0) = 1$ and $0 \le F(x) \le 1$. Let $f_n(x) := F(n(x - x_{n_0}))$ if $x \in \left(x_{n_0} - \frac{\epsilon}{n}, x_{n_0} + \frac{\epsilon}{n}\right)$ and $f_n(x) := 0$ if $x \in (a,b)\backslash\left(x_{n_0} - \frac{\epsilon}{n}, x_{n_0} + \frac{\epsilon}{n}\right)$. Then $f_n \in D(H_{0,M})$ and

$$\|f_n\|^2 = \int_a^b f_n^2(x)\,dx \le \int_a^b f_n(x)\,dx = \int_{x_{n_0}-\frac{\epsilon}{n}}^{x_{n_0}+\frac{\epsilon}{n}} F(n(x - x_{n_0}))\,dx = \int_{-\epsilon}^{\epsilon} \frac{F(y)}{n}\,dy = \frac{1}{n}\int_{-\epsilon}^{\epsilon} F(y)\,dy \longrightarrow 0$$

as $n \to \infty$. Then φ is not bounded because if it were, we would have

$$1 = |f_n(x_{n_0})| \le C\|f_n\| \longrightarrow 0 \qquad \text{as } n \to \infty$$

getting a contradiction.

ii) Let $G(x, y; z)$ be the Green's function defined in Definition 3.5, then

$$\left((H_{0,M} - i)^{-1} f\right)(x) = \int_a^b G(x, y; i) f(y)\, dy.$$

Hence

$$|l(f)| = \left|\left((H_{0,M} - i)^{-1} f\right)(x_{n_0})\right| = \left|\int_a^b G(x_{n_0}, y; i) f(y)\, dy\right| \le \|G(x_{n_0}, \cdot\,; i)\|\, \|f\|.$$

□

Let $\dot{H}_{0,M} = H_{0,M}|_{D_\varphi}$ be the restriction of $H_{0,M}$ to

$$D_\varphi := \{f \in D(H_{0,M}) \,:\, \varphi(f) = f(x_{n_0}) = 0\}. \tag{3.5}$$

From Lemmas 2.2, 2.3 and 3.7, we have that $\dot{H}_{0,M}$ is a symmetric operator with defect indices (1,1). By the von Neumann theory, the selfadjoint extensions T_θ of the symmetric operator $\dot{H}_{0,M}$ are given by

$$D(T_\theta) = \{g + c\psi_+ + ce^{i\theta}\psi_- \,:\, c \in \mathbb{C}, g \in D(\dot{H}_{0,M})\},$$
$$T_\theta(g + c\psi_+ + ce^{i\theta}\psi_-) = \dot{H}_{0,M}\, g + ic\psi_+ - ice^{i\theta}\psi_-$$

for $\theta \in [0, 2\pi)$, where $\psi_\pm \in \mathrm{Kern}(\dot{H}^*_{0,M} \mp i)$.

Lemma 3.8. *The functions $\psi_\pm$ introduced above can be chosen as*

$$\psi_\pm = G(x_{n_0}, \cdot\,; \mp i).$$

Proof. From (3.4), we get

$$\varphi\left((H_{0,M} + i)^{-1} g\right) = \int_a^b G(x_{n_0}, y; i) g(y)\, dy = \int_a^b \overline{G(x_{n_0}, y; -i)} g(y)\, dy = \langle G(x_{n_0}, \cdot\,; -i), g\rangle.$$

Now for $g \in D(\dot{H}_{0,M}) = D_\varphi$,

$$\langle G(x_{n_0}, \cdot\,; -i), (\dot{H}_{0,M} + i) g\rangle = \varphi\left((H_{0,M} + i)^{-1} (\dot{H}_{0,M} + i) g\right) = \varphi(g) = 0.$$

Therefore $G(x_{n_0}, \cdot\,; -i) \in \mathrm{Rang}(\dot{H}_{0,M} + i)^\perp = \mathrm{Kern}(\dot{H}^*_{0,M} - i)$ and analogously $G(x_{n_0}, \cdot\,; i) \in \mathrm{Kern}(\dot{H}^*_{0,M} + i)$. □

Theorem 3.9. *Let T_θ be a selfadjoint extension of $\dot{H}_{0,M}$, then there exists a unique $\alpha \in \mathbb{R} \cup \{\infty\}$ such that $T_\theta = H_{\alpha,M}$. Conversely, given $\alpha \in \mathbb{R} \cup \{\infty\}$ there exists a unique $\theta \in [0, 2\pi)$ such that $T_\theta = H_{\alpha,M}$.*

Proof. We shall prove that given $\theta \in [0, 2\pi)$, there exists $\alpha \in \mathbb{R} \cup \{\infty\}$, and given α there exists θ, such that $T_\theta \subset H_{\alpha,M}$ and since T_θ and $H_{\alpha,M}$ are selfadjoint, the result will follow.

Let us first show that $D(T_\theta) \subset D(H_{\alpha,M})$. If $f \in D(T_\theta)$, using the representation $f = g + c\psi_+ + ce^{i\theta}\psi_-$ and the explicit form for $\psi_\pm$ given in Lemma 3.8, from the properties of u_a and u_b as solutions of $(\tau_{0,M} \pm i) = 0$ we get that f is continuous in (a, b) and $D(T_\theta) \subset \{f \in L^2(a, b) \,:\, f, f' \text{ abs. cont. in } (a, b)\backslash M, -f'' + Vf \in L^2(a, b)\}$.

Let us consider $x_n \in M\backslash\{x_{n_0}\}$. If $f \in D(T_\theta)$ then

$$f'(x_n+) - f'(x_n-) = \alpha_n f(x_n). \tag{3.6}$$

To see this assume $x_n < x_{n_0}$ then for $x_n \in M\backslash\{x_{n_0}\}$ and using (3.3)

$$\begin{aligned}\psi'_\mp(x_n+) - \psi'_\mp(x_n-) &= G'(x_{n_0}, x_n+; \pm i) - G'(x_{n_0}, x_n-; \pm i) = W^{-1}u_b(x_{n_0}, \pm i)(u'_a(x_n+, \pm i) - u'_a(x_n-, \pm i)) \\ &= \alpha_n W^{-1} u_b(x_{n_0}, \pm i) u_a(x_n, \pm i) = \alpha_n G(x_{n_0}, x_n; \pm i)\end{aligned}$$

and analogously if $x_n > x_{n_0}$. From this it follows that every element of $D(T_\theta)$ satisfies (3.6).

Observe that for some $\theta_0 \in [0, 2\pi)$, $H_{0,M} = T_\theta$, with

$$D(T_{\theta_0}) = \{g + c\psi_+ + ce^{i\theta_0}\psi_- : c \in \mathbb{C}, g \in D(\dot{H}_{0,M})\}.$$

If $\psi_+(x_{n_0}) = 0$, then $\psi_-(x_{n_0}) = \overline{\psi_+(x_{n_0})} = 0$ and $D(T_{\theta_0}) = D(H_{0,M}) = D(\dot{H}_{0,M})$ which is impossible since $\dot{H}_{0,M}$ is not selfadjoint. Hence $\psi_+(x_{n_0}) \neq 0$.

Now we will prove that if $f \in D(T_\theta)$, then

$$f'(x_{n_0}+) - f'(x_{n_0}-) = \alpha f(x_{n_0}) \tag{3.7}$$

for some $\alpha \in \mathbb{R}$. Assume that $(g + c\psi_+ + ce^{i\theta}\psi_-)(x_{n_0}) \neq 0$. Let $\alpha \in \mathbb{C}$ be such that

$$(g + c\psi_+ + ce^{i\theta}\psi_-)'(x_{n_0}+) - (g + c\psi_+ + ce^{i\theta}\psi_-)'(x_{n_0}-) = \alpha(g + c\psi_+ + ce^{i\theta}\psi_-)(x_{n_0}). \tag{3.8}$$

Let us verify that α does not depend on g or c and $\alpha \in \mathbb{R}$. By definition of $D(\dot{H}_{0,M})$, we have $g'(x_{n_0}+) = g'(x_{n_0}-)$ and $g(x_{n_0}) = 0$, therefore the equality (3.8) becomes

$$c(\psi_+ + e^{i\theta}\psi_-)'(x_{n_0}+) - c(\psi_+ + e^{i\theta}\psi_-)'(x_{n_0}-) = c\alpha(\psi_+ + e^{i\theta}\psi_-)(x_{n_0})$$

and

$$\alpha = \frac{(\psi_+ + e^{i\theta}\psi_-)'(x_{n_0}+) - (\psi_+ + e^{i\theta}\psi_-)'(x_{n_0}-)}{(\psi_+ + e^{i\theta}\psi_-)(x_{n_0})}.$$

In fact, using the explicit form $\psi_\pm = G(x_{n_0}, \cdot; \mp i)$ we get

$$\alpha = -\frac{1 + e^{i\theta}}{\psi_+(x_{n_0}) + e^{i\theta}\overline{\psi_+(x_{n_0})}}.$$

Note that $\psi_-(x_{n_0}) = \overline{\psi_+(x_{n_0})}$, where $\overline{z}$ denotes the complex conjugate of z. Since $\alpha = \overline{\alpha}$ we get $\alpha \in \mathbb{R}$.

For the other direction, if we are given $\alpha \in \mathbb{R}$, then we take $\theta \in [0, 2\pi)$ such that

$$-\frac{1 + \alpha\psi_+(x_{n_0})}{1 + \alpha\overline{\psi_+(x_{n_0})}} = e^{i\theta} \tag{3.9}$$

and reversing the argument above we get Equation (3.8). Then we have $D(T_\theta) \subset D(T_{\alpha,M})$.

The case $(g + c\psi_+ + ce^{i\theta}\psi_-)(x_{n_0}) = 0$ happens when θ is such that

$$e^{i\theta} = -\frac{\psi_+(x_{n_0})}{\psi_-(x_{n_0})}$$

so this corresponds to taking the limit $\alpha \to \infty$ in formula (3.9).

Since u_a, u_b and g satisfy the condition $[v, f]_a = [w, f]_b = 0$, then these conditions are satisfied for all $f \in D(T_\theta)$, where v, w are solutions of $(\tau_{0,M} - \lambda)u = 0$, $\lambda \in \mathbb{R}$, near a and near b respectively. Since we can construct solutions of $(\tau_{\alpha,M} - \lambda)u = 0$ which coincide with v and w near a and b respectively, then $f \in D(T_\theta)$ satisfies also the conditions needed in the lcc for $H_{\alpha,M}$. Then $D(T_\theta) \subset D(H_{\alpha,M})$.

Now, we claim that

$$T_\theta f = H_{\alpha,M} f = \tau_{\alpha,M} f, \qquad \text{for all } f \in D(T_\theta). \tag{3.10}$$

Since $f \in D(T_\theta)$, it has the representation $f = g + c\psi_+ + ce^{i\theta}$ and $T_\theta f = \dot{H}_{0,M} g + ic\psi_+ - ice^{i\theta}\psi_-$. By definition of u_a and u_b we have

$$(\tau_{\alpha,M} - i)\psi_+(y) = (\tau_{\alpha,M} - i)u_a(x_{n_0})u_b(y) = 0, \quad y \geq x_{n_0},$$

$$(\tau_{\alpha,M} - i)\psi_+(y) = (\tau_{\alpha,M} - i)u_b(x_{n_0})u_a(y) = 0, \quad y < x_{n_0}.$$

And $\tau_{\alpha,M}\psi_+ = i\psi_+ = \dot{H}^*_{0,M}\psi_+$, analogously for ψ_-. Moreover, $\dot{H}_{0,M} g = \tau_{0,M} g = \tau_{\alpha,M} g$, for all $\alpha \in \mathbb{R}$, since g' is continuous at x_{n_0} and $g(x_{n_0}) = 0$, therefore the claim (3.10) follows.

We have proved that $T_\theta \subset H_{\alpha,M}$. The selfadjointness implies $T_\theta = H_{\alpha,M}$.

Since $D(T_\theta) \neq D(T_{\theta'})$ and $D(H_{\alpha,M}) \neq D(H_{\alpha',M})$, for $\theta \neq \theta'$ and $\alpha \neq \alpha'$, then for α there corresponds a unique θ and conversely. □

Remark 3.10. Note that $\alpha = 0$ corresponds to $\theta = \pi$, that is $T_\pi = H_{0,M}$.

Remark 3.11. Note that in the case that θ corresponds to $\alpha = \infty$ it happens that $D(T_\theta) = \text{Kern}(\tilde{\varphi})$, where $\tilde{\varphi} : D(T_\theta) \to \mathbb{C}$, $\tilde{\varphi}(f) = f(x_{n_0})$. This does not conflict with Lemma 2.3 because in this case $\tilde{\varphi}$ is continuous in $D(T_\theta)$.

Using the above results we can prove the following statement, which is analogous to Theorem 2.4.

Theorem 3.12. *Let $E \in \mathbb{R}$ be fixed. Then for the set*

$$A(E) := \{\alpha \in \mathbb{R} : E \in \sigma_p(H_{\alpha,M})\}$$

there are two possibilities:

i) $A(E) = \mathbb{R}$.
ii) *$A(E)$ has at most one element.*

Proof. Suppose i) does not hold, then there exists $\alpha \in \mathbb{R}$ such that $\alpha \notin A(E)$. From Theorem 3.9, there exists $\theta \in [0, 2\pi)$ such that $T_\theta = H_{\alpha,M}$, where T_θ is a selfadjoint extension of $\dot{H}_{0,M}$. Since E is not an eigenvalue of T_θ, we are in case ii) of Theorem 2.1 and E is an eigenvalue of T_{θ_0} for at most one $\theta_0 \in [0, 2\pi)$ and therefore E is an eigenvalue of $H_{\alpha_0,M}$ for at most one $\alpha_0 \in \mathbb{R}$. □

Remark 3.13. From Theorem 2.1, case i) happens if and only if the eigenvector associated to the eigenvalue E is on $D(\dot{H}_{0,M})$.

4 | RANDOM STURM–LIOUVILLE OPERATORS WITH δ-POINT INTERACTIONS

In this section we use the previously obtained results to study the random case. First the probability space Ω where the sequences of coupling constants live is constructed and then our random operators are defined.

The space of real valued sequences $\{\omega_n\}_{n\in I}$, where $I \subseteq \mathbb{Z}$, will be denoted by $\mathbb{R}^I$. We introduce a measure in $\mathbb{R}^I$ in the following way. Let $\{p_n\}_{n\in I}$ be a sequence of continuous probability measures in $\mathbb{R}$ ($p_n(\{r\}) = 0$ for any $r \in \mathbb{R}$) and consider the product measure $\mathbb{P} = \times_{n\in I} p_n$ defined on the product σ-algebra $\mathcal{F}$ of $\mathbb{R}^I$ generated by the cylinder sets, that is, by the sets of the form $\{\omega : \omega(i_1) \in A_1, \dots, \omega(i_n) \in A_n\}$ for $i_1, \dots, i_n \in I$, where $A_1, \dots, A_n$ are Borel sets in $\mathbb{R}$. In this way a measure space $\tilde{\Omega} = (\mathbb{R}^I, \mathcal{F}, \mathbb{P})$ is constructed. We consider then the completion of this space (subsets of sets of measure zero are measurable) $\tilde{\Omega}$ which will be denoted by Ω. See Chapter 1, Section 1 in [20].

Let $-\infty \le a < b \le \infty$, let $V \in L^1_{loc}(a,b)$ be a real valued function. Fix a discrete set $M := \{x_n\}_{n\in I} \subset (a,b)$ where $I \subseteq \mathbb{Z}$ and let $\omega = \{\omega(n)\}_{n\in I} \in \Omega$. Consider the formal differential expression

$$\tau_\omega := -\frac{d^2}{dx^2} + V(x) + \sum_{n\in I} \omega(n)\delta(x - x_n).$$

The maximal operator T_ω corresponding to τ_ω is defined as before by

$$T_\omega f = \tau f,$$

$$D(T_\omega) = \{f \in L^2(a,b) : f, f' \text{ abs. cont. in } (a,b)\backslash M, -f'' + Vf \in L^2(a,b), \\ f(x_n+) = f(x_n-), f'(x_n+) - f'(x_n-) = \omega(n)f(x_n), \text{ for all } n \in I\}.$$

Assume the limit point occurs at a or that τ_ω is regular at a (see Definition 3.6) and the same possibilities for b.

For $\theta, \gamma \in [0,\pi)$ fixed, let $H_\omega^{\theta,\gamma}$ be the selfadjoint restriction of T_ω defined as

$$H_\omega^{\theta,\gamma} f = \tau f, \tag{4.1}$$
$$D\left(H_\omega^{\theta,\gamma}\right) = \left\{ f \in D(T_\omega) : \begin{array}{ll} f(a)\cos\theta + f'(a)\sin\theta = 0 & \text{in case } \tau_\omega \text{ regular at } a, \\ f(b)\cos\gamma + f'(b)\sin\gamma = 0 & \text{in case } \tau_\omega \text{ regular at } a \end{array} \right\}.$$

Notice that the index θ or γ are meaningless if τ_ω is lpc at a or b.

In what follows instead of $H_\omega^{\theta,\gamma}$ we shall write H_ω.

Remark 4.1. One example where τ_ω is lpc at both ends for all $\omega \in \Omega$ was given in Theorem 1 [6]. There it was required that $I = \mathbb{Z}$, V bounded and $\inf_{n\in\mathbb{Z}} |x_{n+1} - x_n| > 0$.

The condition $\inf |x_{n+1} - x_n| > 0$ can be significantly relaxed. Or one can assume some sort of semiboundedness instead. See [1, 10 18] and [19].

Definition 4.2. For any $E \in \mathbb{R}$, we define

$$A(E) := \{\omega \in \Omega : E \in \sigma_p(H_\omega)\}. \tag{4.2}$$

For any measurable $B \subseteq A(E)$ and any $n \in I$, define

$$Q_{n,E} := \{\omega \in B \mid \exists\, u_\omega \in D(H_\omega),\ H_\omega u_\omega = E u_\omega \text{ and } u_\omega(x_n) \ne 0\}. \tag{4.3}$$

Lemma 4.3. *$Q_{n,E}$ is measurable and* $\mathbb{P}(Q_{n,E}) = 0$.

Proof. Let

$$\chi_B(\omega) \quad = \quad \begin{cases} 1 & \text{if } \omega \in B, \\ 0 & \text{if } \omega \notin B. \end{cases}$$

If $\omega \in Q_{n,E}$, then from the definition of $Q_{n,E}$ it follows $\chi_B(\omega) = 1$.

Let $f : \mathbb{R}^{I\backslash\{n\}} \to [0,\infty)$.

$$f(\tilde{\omega}) := \int_{\mathbb{R}} \chi_B(\omega)\, dp_n(\omega(n))$$

where $\tilde{\omega} = \sum_{k\in I\setminus\{n\}} \omega(k)e(k)$. Here $e(k) = (e_m)_{m\in I}$ are the canonical vectors with entries $e_m = 0$ if $k \neq m$ and $e_k = 1$. The measurability of f follows from Fubini's theorem. (See Theorem 7.8 [22].)

If $\omega = \sum_{k\in I} \omega(k)e(k) \in Q_{n,E}$ then $f(\tilde{\omega}) = 0$, where $\tilde{\omega} = \sum_{k\in I\setminus\{n\}} \omega(k)e(k)$, since p_n is continuous and from Theorem 3.12.

Hence $Q_{n,E} \subseteq [f^{-1}(\{0\}) \times \mathbb{R}] \cap B$.

Now, using Fubini,

$$\int_{f^{-1}(\{0\})\times\mathbb{R}} \chi_B(\omega)\, d\mathbb{P} = \int_{f^{-1}(\{0\})} d\mathbb{P}(\tilde{\omega}) \int_{\mathbb{R}} \chi_B(\omega)\, dp_n(\omega(n)) = \int_{f^{-1}(\{0\})} f(\tilde{\omega})\, d\mathbb{P}(\tilde{\omega}) = 0.$$

Then,

$$\int_{[f^{-1}(\{0\})\times\mathbb{R}]\cap B} \chi_B(\omega)\, d\mathbb{P} = 0$$

and since $\chi_B(\omega) = 1$ in B, then $\mathbb{P}([f^{-1}(\{0\}) \times \mathbb{R}] \cap B) = 0$.

Since the measure $d\mathbb{P}$ is complete, then any subset of a measurable set of measure zero is measurable with measure zero. Therefore $Q_{n,E}$ is measurable. □

Theorem 4.4. *Let $E \in \mathbb{R}$ be fixed and let B be any measurable subset of $A(E)$. Then one of the following options hold:*

i) $\mathbb{P}(B) = 0$,
ii) $A(E) = \Omega$.

Proof. It will be enough to proof that if *ii*) doesn't hold then *i*) must hold.

Assume then that there exists $\omega_0 \in \Omega$ such that E is not eigenvalue of H_{ω_0}. If E is not eigenvalue of H_ω, for all $\omega \in \Omega$, then $\mathbb{P}(B) = 0$ and the result follows.

Suppose now $\omega \in B$, then $E \in \sigma_p(H_\omega)$, i.e. there exists $u_\omega \in D(H_\omega)$ such that $H_\omega u_\omega = Eu_\omega$. Then $\omega \in Q_{n,E}$, for some $n \in I$. This follows because if $u_\omega(x_n) = 0$ for all $n \in I$, then from the definition of H_ω, E must be an eigenvalue of H_{ω_0}. Therefore

$$B \subset \bigcup_{n\in I} Q_{n,E}.$$

Using Lemma 4.3, then $\mathbb{P}(\bigcup_{n\in I} Q_n) = 0$, therefore the result follows. □

For the next corollary we denote by H the operator H_ω defined in (4.1) with $\omega(n) = 0$, for all $n \in I$. This is just the selfadjoint operator generated by the differential expression τ in the classical Sturm–Liouville theory without point interactions.

Corollary 4.5 (cf. Theorem 3.12).

a) If $E \notin \sigma_p(H)$ then $\mathbb{P}(B) = 0$ for any measurable subset B of $\omega \in \Omega$ for which $E \in \sigma_p(H_\omega)$.
b) If $E \in \sigma_p(H)$ with $Hu = Eu$, then $A(E) = \Omega$ if and only if $u(x_n) = 0$, for all $n \in I$.

Proof.

a) If $E \notin \sigma_p(H)$, then $\omega = (\dots, 0, 0, 0, \dots) \notin A(E)$. Therefore $A(E) \neq \Omega$ and the assertion follows from Theorem 4.4.
b) Suppose $E \in \sigma_p(H)$ with $Hu = Eu$.
   1. $\Leftarrow$) If $u(x_n) = 0$ for all $n \in I$, then from the definition of H_ω, E must be an eigenvalue of H_ω with eigenvector u, for all $\omega \in \Omega$.
   2. $\Rightarrow$) From Lemma 4.3, $\mathbb{P}(\bigcup_{n\in I} Q_{n,E}) = 0$. Then $A(E) = \Omega \not\subseteq \bigcup_{n\in I} Q_{n,E}$.
   Take $\tilde{\omega} \in A(E)\setminus\bigcup_{n\in I} Q_{n,E}$. There exists $u_{\tilde{\omega}} \in D(H_{\tilde{\omega}})$ such that

$$(\tau_{\tilde{\omega}} - E)u_{\tilde{\omega}} = 0 \qquad \text{and} \qquad u_{\tilde{\omega}}(x_n) = 0, \text{ for all } n \in I.$$

Therefore $u'_{\tilde{\omega}}(x_n+) - u'_{\tilde{\omega}}(x_n-) = \tilde{\omega}(n)u_{\tilde{\omega}}(x_n) = 0$, for all $n \in I$. Hence $u'_{\tilde{\omega}}$ is continuous in (a, b) and $(H - E)u_{\tilde{\omega}} = 0$.
Since all eigenvalues of H are simple, see Theorem 8.29 (d) [25], then $u(x_n) = Cu_{\tilde{\omega}}(x_n) = 0$, for all $n \in I$.

Then unless E is an eigenvalue of H and the point interactions are placed at the roots of eigenfunctions, we will have a "small" set of operators H_ω sharing the same eigenvalue E.

As another consequence of Theorem 4.4 we get the following corollary.

Corollary 4.6. *Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of real numbers and let B_i be measurable subsets of $A(E_i)$. Assume there is no point $E \in \mathbb{R}$ which is eigenvalue of H_ω for all $\omega \in \Omega$, then*

$$\mathbb{P}\left(\left\{\omega \in \bigcup_{i=1}^{\infty} B_i \ : \ \{E_i\}_{i=1}^{\infty} \cap \sigma_p(H_\omega) \neq \emptyset\right\}\right) = 0.$$

Proof. By additivity of $\mathbb{P}$ and Theorem 4.4, we have

$$\begin{aligned}
&\mathbb{P}\left(\{\omega \in \Omega : \{E_i\}_{i=1}^{\infty} \cap \sigma_p(H_\omega) \neq \emptyset\}\right) \\
&= \mathbb{P}\left(\left\{\omega \in \Omega : \bigcup_{i=1}^{\infty} [\{E_i\} \cap \sigma_p(H_\omega)] \neq \emptyset\right)\right\} \\
&= \mathbb{P}\left(\bigcup_{i=1}^{\infty} \{\omega \in \Omega : E_i \in \sigma_p(H_\omega)\}\right) \\
&\leq \sum_{i=1}^{\infty} \mathbb{P}(\{\omega \in \Omega : E_i \in \sigma_p(H_\omega)\}) = 0.
\end{aligned}$$

□

4.1 | Oscillation of solutions

We shall use results about the oscillation of solutions of second order differential expressions. The location of zeros of eigenfunctions together with knowledge about the positions of the point interactions, will help us to understand when option b) in Theorem 4.4 happens.

In this subsection τ is as in Equation (3.1) and $A(E)$ is as in Definition 4.2 (4.2), that is the set of $\omega \in \Omega$ such that H_ω share the common eigenvalue E.

Definition 4.7 See Section XI.6 in [12]**.** The equation

$$(\tau - E)f = 0$$

is said to be *nonoscillatory on an interval* J if every solution has at most a finite number of zeros on J.

If $t = b$ is an (possibly infinite) endpoint of J which does not belong to J, then the equation is said to be *nonoscillatory at* $t = b$ if every solution has a finite number of zeros in J or if the zeros do not accumulate at b.

Lemma 4.8. *If $A(E) = \Omega$, then there exists a solution u of $(\tau - E)f = 0$ such that $u(x_n) = 0$, for all $n \in I$.*

Proof. If $A(E) = \Omega$, then there exists u such that $Hu = Eu$, where H is the operator H_ω with $\omega(n) = 0$, for all $n \in I$. From Corollary 4.5 (b) the assertion follows. □

Theorem 4.9. *Let V be the potential appearing in the expression (3.1). Assume $|V(x)| \leq K$ for all $x \in (a, b)$. Let J be an interval such that*

$$|J| \leq \frac{2}{\sqrt{K + |E|}}$$

where $|J|$ denotes the length of the interval. Assume $J \cap M$ has at least two elements. Then $\mathbb{P}(B) = 0$ for any measurable subset B of $A(E)$.

Proof. Suppose there exists a measurable subset B of $A(E)$ such that $\mathbb{P}(B) > 0$, then from Theorem 4.4, $A(E) = \Omega$.

By Lemma 4.8, there exists a solution u of $(\tau - E)f = 0$ such that $u(x_n) = 0$, for all $n \in I$. Using a theorem due to Lyapunov, see Theorem 3.9 of [14] and Corollary 5.1 of [12], the interval J is disconjugate, i.e. there is at most one zero of any solution of $(\tau - E)f = 0$ in the interval J, since u is solution this is a contradiction, hence $\mathbb{P}(B) = 0$ for any measurable subset B of $A(E)$. □

In the last theorem observe that the larger $|E|$ is the smaller $|J|$ has to be. This corresponds to the fact that the solutions oscillate faster if the energy is high.

Theorem 4.10. *Suppose $(\tau - E)f = 0$ is nonoscillatory in (a, b) and the set of interactions M is a countable set. Then $\mathbb{P}(B) = 0$ for any measurable subset B of $A(E)$.*

Proof. Suppose there exists a measurable subset B of $A(E)$ such that $\mathbb{P}(B) > 0$, then from Theorem 4.4, $A(E) = \Omega$.

By Lemma 4.8, there exists a solution u of $(\tau - E)f = 0$ such that $u(x_n) = 0$, for all $n \in I$. The equation $(\tau - E)f = 0$ is nonoscillatory i.e. any solution in the interval (a, b) has at most a finite number of zeros. Since u is solution this is a contradiction, hence $\mathbb{P}(B) = 0$ for any measurable subset B of $A(E)$. □

There are several conditions in the literature which allow us to conclude that our problem is nonoscillatory. Applying a theorem of Hille, [14, Theorem 3.1], we get the following result.

Theorem 4.11. *If V is continuous in $[a, \infty)$, $V(x) \le E$, $\int_a^\infty (E - V(x))\,dx < \infty$ and*

$$\limsup_{x\to\infty} x \int_x^\infty (E - V(t))\,dt < \frac{1}{4},$$

then $(\tau - E)f = 0$ is nonoscillatory at $[a, \infty)$.

Finer estimates on the number of zeros can be used too, as the following result shows.

Theorem 4.12. *Let V be continuous in $[0, T]$ satisfying $|V(x)| \le K$. If the number of points in $M \cap [0, T]$ is greater or equal to*

$$\frac{T\sqrt{|E| + K}}{2} + 1,$$

then $\mathbb{P}(B) = 0$ for any measurable subset B of $A(E)$.

Proof. Using Corollary 5.2 in [12] we see that the number of zeros of any solution of $(\tau - E)f = 0$ is less than

$$\frac{T\sqrt{|E| + K}}{2} + 1$$

and then the proof follows as in Theorem 4.9. □

4.2 | Measurable operators

Now we introduce condition of measurability for the family of operators H_ω.

Definition 4.13. See Lemma 1.2.2 in [24], Proposition 3 in [16]. A family $\{S_\omega\}_{\omega\in\Omega}$ of selfadjoint operators in a Hilbert space $\mathfrak{H}$ is called *measurable* if the mappings

$$\omega \to \langle \varphi, E_\omega(\lambda)\psi\rangle$$

are measurable for all φ, $\psi \in \mathfrak{H}$, where $E_\omega(\lambda)$ is the corresponding resolution of the identity of S_ω.

Theorem 4.14. [Communicated to us by Peter Stollmann] *Let*

$$A(E) := \{\omega \in \Omega : E \in \sigma_p(S_\omega)\}$$

be as in Definition 4.2. If $\{S_\omega\}_{\omega\in\Omega}$ is a measurable family of operators defined in a separable Hilbert space $\mathfrak{H}$, then $A(E)$ is measurable.

Proof. Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\mathfrak{H}$.
Observe that

$$A(E) = \bigcup_{n\in\mathbb{N}} A_n \tag{4.4}$$

where

$$A_n := \{\omega \in \Omega : E_\omega(\{E\})\psi_n \neq 0\}.$$

The set on the right hand side of (4.4) is contained in $A(E)$ since $A(E) = \{\omega \in \Omega \,|\, E_\omega(\{E\}) \neq 0\}$. To prove the other inclusion, let $\omega \in A(E)$ and assume that for all n, $E_\omega(\{E\})\psi_n = 0$. For any $x \in \mathfrak{H}$ we have

$$\langle E_\omega(\{E\})x, \psi_n\rangle = \langle x, E_\omega(\{E\})\psi_n\rangle = 0.$$

Since $\{\psi_n\}$ is dense, $E_\omega(\{E\})x = 0$ and $E_\omega(\{E\}) = 0$, which is a contradiction to $\omega \in A(E)$. Therefore there is n_0 such that $E_\omega(\{E\})\psi_{n_0} \neq 0$ and $\omega \in \bigcup_{n\in\mathbb{N}} A_n$.

We shall now prove that the sets A_n are measurable.

Since S_ω is measurable, the function f_n defined as $\omega \to f_n(\omega) := \langle \psi_n, E_\omega(\{E\})\psi_n\rangle$ is measurable for each n. We get $\omega \in A_n^c$ if and only if

$$f_n(\omega) = \langle \psi_n, E_\omega(\{E\})\psi_n\rangle = \|E_\omega(\{E\})\psi_n\|^2 = 0.$$

Thus

$$A_n^c = \{\omega \,|\, E_\omega(\{E\})\psi_n = 0\} = f_n^{-1}(\{0\}).$$

It follows that A_n^c and therefore A_n are measurable sets. Hence $A(E)$ is a countable union of measurable sets, thus measurable. □

Using Theorem 4.14 we obtain the following corollary of Theorem 4.4. Let $\{H_\omega\}_{\omega\in\Omega}$ be the family of operators introduced in (4.1).

Corollary 4.15. *Assume that the family $\{H_\omega\}_{\omega\in\Omega}$ is measurable. For fixed $E \in \mathbb{R}$, one of the following options hold:*

i) $\mathbb{P}(A(E)) = 0$;
ii) $A(E) = \Omega$.

Proof. Take $B = A(E)$ in Theorem 4.4. □

As an example of a measurable family, let us mention the operators generated by the formal differential expression

$$\tau_\omega := -\frac{d^2}{dx^2} + \sum_{n\in I} \omega(n)\delta(x - x_n)$$

where $\omega(n)$ is a stationary metrically transitive random field satisfying $|\omega(n)| \le C < \infty$, see [16]. In particular we can take $\omega(n)$ to be independent identically distributed random variables.

Since the operator generated by $-\frac{d^2}{dx^2}$ without point interactions does not have eigenvalues, we can apply Corollary 4.5 and obtain $\mathbb{P}(A(E)) = 0$. We get in this case a proof of a result due to Pastur which says that the probability of any fixed $\lambda \in \mathbb{R}$ being an eigenvalue of finite multiplicity of a metrically transitive operator is zero, see Theorem 3 in [21] and Theorem 2.12 in [20].

5 | STURM–LIOUVILLE OPERATORS WITH δ'-POINT INTERACTIONS

Now we consider operators with δ'-interactions and show how analogous results can be obtained. Let $-\infty \le a < b \le \infty$, let $V \in L^1_{loc}(a, b)$ be a real valued function. Fix a discrete set $M := \{x_n\}_{n\in I} \subset (a, b)$ where $I \subseteq \mathbb{Z}$ and let $\{\alpha_n\}_{n\in I} \subset \mathbb{R}$.

Consider the formal differential expression

$$\tau_{\alpha,M} := -\frac{d^2}{dx^2} + V(x) + \sum_{n\in I} \alpha_n \delta'(x - x_n).$$

The maximal operator $T_{\alpha,M}$ corresponding to $\tau_{\alpha,M}$ is defined by

$$T_{\alpha,M} f = \tau f = -\frac{d^2 f}{dx^2} + Vf,$$

$$D(T_{\alpha,M}) = \{f \in L^2(a, b) : f, f' \text{ abs. cont. in } (a, b)\backslash M, -f'' + Vf \in L^2(a, b), \\ f'(x_n+) = f'(x_n-), f(x_n+) - f(x_n-) = \alpha_n f'(x_n), \text{ for all } n \in I\}.$$

The construction is similar to that we have done in Section 4, but notice the change of the conditions at the points x_n.

Definition 5.1. A function f is a *solution of* $(\tau_{\alpha,M} - \lambda)f = 0$ if f and f' are absolutely continuous in $(a, b)\backslash M$ with $-f'' + Vf - \lambda f = 0$ and $f'(x_n+) = f'(x_n-)$, $f(x_n+) - f(x_n-) = \alpha_n f'(x_n)$, for all $n \in I$.

Consider the selfadjoint restriction $H_{\alpha,M}$ of $T_{\alpha,M}$ in $L_2(a, b)$ defined as

$$H_{\alpha,M} f = \tau f, \qquad (5.1)$$
$$D(H_{\alpha,M}) = \left\{ f \in D(T_{\alpha,M}) : \begin{array}{l} [v, f]_a = 0 \text{ if } \tau_{\alpha,M} \text{ lcc at } a, \\ [w, f]_b = 0 \text{ if } \tau_{\alpha,M} \text{ lcc at } b \end{array} \right\},$$

where v and w are non-trivial real solutions of $(\tau_{\alpha,M} - \lambda)v = 0$ near a and near b respectively, $\lambda \in \mathbb{R}$. See [5, Theorem 5.2].

For the next definition we take $\alpha = 0$.

Definition 5.2. Let us define the function $\mathcal{G}(x, y; z)$ for $H_{0,M}$ as

$$\mathcal{G}(x, y; z) := \begin{cases} W(u_b, u_a)^{-1} u_a'(x, z) u_b(y, z) & \text{if } x \le y, \\ W(u_b, u_a)^{-1} u_b'(x, z) u_a(y, z) & \text{if } x > y, \end{cases} \qquad (5.2)$$

where u_a and u_b are solutions of $(\tau_{0,M} - z)u = 0$ (see Definition 5.1) with $[v, u_a]_a = 0$ if $\tau_{0,M}$ lcc at a and $[w, u_b]_b = 0$ if $\tau_{0,M}$ lcc at b.

Let us take from now on $\varphi : D(H_{0,M}) \to \mathbb{C}$ given by $\varphi(f) := \delta'_{x_{n_0}}(f) = f'(x_{n_0})$. Similar to Lemma 3.7 we have:

Lemma 5.3.

i) *The functional φ is not continuous in $D(H_{0,M})$ with the norm of $L^2(a,b)$.*
ii) *The functional $l : L^2(a,b) \to \mathbb{C}$ defined as $l(f) := \varphi\left((H_{0,M} + i)^{-1} f\right)$ is continuous.*

Proof.

i) Take $\epsilon > 0$ such that $I_\epsilon \cap M = \{x_{n_0}\}$, where $I_\epsilon = [x_{n_0} - \epsilon, x_{n_0} + \epsilon]$. Let $F \in C_0^\infty(-\epsilon, \epsilon)$ such that $F'(0) = 1$ and $0 \le F(x) \le 1$. Let $f_n(x) := F(n(x - x_{n_0}))$ if $x \in \left(x_{n_0} - \frac{\epsilon}{n}, x_{n_0} + \frac{\epsilon}{n}\right)$ and $f_n(x) := 0$ if $x \in (a,b) \backslash \left(x_{n_0} - \frac{\epsilon}{n}, x_{n_0} + \frac{\epsilon}{n}\right)$. Then $f_n \in D(H_{0,M})$ and

$$\|f_n\|^2 = \int_a^b f_n^2(x)\,dx \le \int_a^b f_n(x)\,dx = \int_{x_{n_0}-\frac{\epsilon}{n}}^{x_{n_0}+\frac{\epsilon}{n}} F(n(x - x_{n_0}))\,dx = \frac{1}{n}\int_{-\epsilon}^{\epsilon} F(y)\,dy \longrightarrow 0$$

as $n \to \infty$. Then φ is not bounded because if it would be, we would have

$$n = |nF'(0)| = |f_n'(x_{n_0})| \le C\|f_n\| \longrightarrow 0 \quad \text{as } n \to \infty$$

getting a contradiction.

ii) Let $G(x, y; z)$ be the Green's function defined in Definition 3.5, then

$$\begin{aligned} \frac{d}{dx}\left((H_{0,M} - i)^{-1} f\right)(x) &= u_b'(x)\int_a^x u_a(y) f(y)\,dy + u_a'\int_x^b u_b(y) f(y)\,dy \\ &= \int_a^b \mathcal{G}(x, y; i) f(y)\,dy \end{aligned}$$

Hence

$$|l(f)| = \left|\left((H_{0,M} - i)^{-1}\right) f(x_{n_0})\right| = \left|\int_a^b \mathcal{G}(x_{n_0}, y; i) f(y)\,dy\right| \le \|\mathcal{G}(x_{n_0}, \cdot\,; i)\|\,\|f\|.$$

□

Let $\dot{H}_{0,M} = H_{0,M}|_{D_\varphi}$ with

$$D_\varphi := \{f \in D(H_{0,M}) : \varphi(f) = f'(x_{n_0}) = 0\}. \tag{5.3}$$

From Lemmas 2.2, 2.3 and 5.3, we have that $\dot{H}_{0,M}$ is a symmetric operator with defect indices (1,1). By the von Neumann theory, the selfadjoint extensions T_θ of the symmetric operator $\dot{H}_{0,M}$ are given by

$$\begin{aligned} D(T_\theta) &= \{g + c\psi_+ + ce^{i\theta}\psi_- : c \in \mathbb{C}, g \in D(\dot{H}_{0,M})\}, \\ T_\theta(g + c\psi_+ + ce^{i\theta}\psi_-) &= \dot{H}_{0,M}\, g + ic\psi_+ - ice^{i\theta}\psi_-, \end{aligned}$$

for $\theta \in [0, 2\pi)$, where $\psi_\pm \in \mathrm{Kern}(\dot{H}^*_{0,M} \mp i)$.

Lemma 5.4. *The functions $\psi_\pm$ introduced above can be chosen as*

$$\psi_\pm = \mathcal{G}(x_{n_0}, \cdot; \mp i).$$

Proof. The proof is similar to that of Lemma 3.8. □

Theorem 5.5. *Let T_θ be a selfadjoint extension of $\dot{H}_{0,M}$. Then there exists a unique $\alpha \in \mathbb{R}$ such that $T_\theta = H_{\alpha,M}$. Conversely, given $\alpha \in \mathbb{R}$ there exists a unique $\theta \in [0, 2\pi)$ such that $T_\theta = H_{\alpha,M}$.*

Proof. The proof is similar to that of Theorem 3.9. □

Theorem 5.6. *Let $E \in \mathbb{R}$ be fixed. Then for the set*

$$A(E) := \{\alpha \in \mathbb{R} : E \in \sigma_p(H_{\alpha,M})\}$$

there are two possibilities:

i) $A(E) = \mathbb{R}$.
ii) $A(E)$ *has at most one element.*

Proof. The proof is similar to that of Theorem 3.12. □

Remark 5.7. From Theorem 2.1, case *i*) happens if and only if the eigenvector associated to the eigenvalue E is on $D(\dot{H}_{0,M})$.

Let $\omega = \{\omega(n)\}_{n\in I} \in \Omega$, where Ω is defined as in Section 4. Consider the formal differential expression

$$\tau_\omega := -\frac{d^2}{dx^2} + V(x) + \sum_{n\in I} \omega(n)\delta'(x - x_n).$$

The maximal operator T_ω corresponding to τ_ω is defined by

$$T_\omega f = \tau f = -\frac{d^2 f}{dx^2} + V f,$$

$$D(T_\omega) = \{f \in L^2(a, b) : f, f' \text{ abs. cont. in } (a, b)\backslash M, -f'' + V f \in L^2(a, b), \\ f'(x_n+) = f'(x_n-), f(x_n+) - f(x_n-) = \omega(n) f'(x_n), \text{ for all } n \in I\}.$$

The construction is similar to that we have done in Section 4, but notice the change of the conditions at the points x_n.

Assume the limit point occurs at a or that τ_ω is regular at a (see Definition 3.6) and the same possibilities for b.

For $\theta, \gamma \in [0, \pi)$ fixed, let $H_\omega^{\theta,\gamma}$ be the selfadjoint restriction of T_ω defined as

$$H_\omega^{\theta,\gamma} f = \tau f,$$
$$D(H_\omega^{\theta,\gamma}) = \left\{ f \in D(T_\omega) : \begin{matrix} f(a)\cos\theta + f'(a)\sin\theta = 0 & \text{in case } \tau_\omega \text{ regular at } a, \\ f(b)\cos\gamma + f'(b)\sin\gamma = 0 & \text{in case } \tau_\omega \text{ regular at } a \end{matrix} \right\}. \tag{5.4}$$

Notice that the index θ or γ are meaningless if τ_ω is lpc at a or b.

In what follows instead of $H_\omega^{\theta,\gamma}$ we shall write H_ω.

Similarly to what has been done before, one can prove for this H_ω with δ' interactions the following theorem.

Theorem 5.8. *Let $E \in \mathbb{R}$ be fixed and let B be any measurable subset of*

$$A(E) := \{\omega \in \Omega : E \in \sigma_p(H_\omega)\}.$$

Then one of the following options hold:

i) $\mathbb{P}(B) = 0$,
ii) $A(E) = \Omega$.

Remark 5.9. Mixed situations where δ and δ' interactions are present can be treated with the arguments given above.

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